Loan Guarantees
Part III - The Revised BSOPM - Model Calibration

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In Part III to the series on Loan Guarantees we will calibrate the continuous time guarantee valuation model that we built in Part II of this series.

Our Hypothetical Problem

We will continue to use the hypothetical problem from Part I...

Table 1: Model Parameters

Symbol	Description	Value	Reference
A_0	Enterprise value at time zero	1,366,700	From Part I
C_0	Annualized cash flow at time zero	100,000	Hypothetical problem
D_T	Debt payoff amount at time T	500,000	Hypothetical problem
T	Guarantee term in years	3.0000	Hypothetical problem
α	Risk-free rate (continuous time)	0.0392	From Part I
κ	Cost of capital (continuous time)	0.0979	From Part I
μ	Cash flow growth rate (continuous time)	0.0247	From Part I
p	Cumulative probability of default	0.1000	Hypothetical problem
ϕ	Dividend yield (continuous time)	0.0732	From Part I
π	Recovery rate given default	0.4000	Hypothetical problem

We will use our model to answer the following question:

Question 1: What is the value of the volatility parameter σ?

Question 2: What is the value of the recovery given default parameter Γ?

Model Calibration - Volatility

In Part II we defined the variable A_t to be enterprise value at time t, the variable κ to be the cost of capital, the variable ϕ to be dividend yield, and the variable σ to be volatility. The equation for enterprise value at time t from the perspective of time zero is...

$$A_t = A_0 \operatorname{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\,Z\right\} \text{ ...where... } Z \sim N\left[0,1\right] \tag{1}$$

In Part II we defined the variable D_t to be the debt payoff amount at time t and the variable a to be the default point, which is the value of the random variable Z in Equation (1) above such that enterprise value at time t is equal to the debt payoff amount at time t. The equation for the default point was...

$$\text{if... } A_t = D_t \text{ ...then... } a = \left[\ln\left(\frac{D_t}{A_0}\right) - \left(\kappa - \phi - \frac{1}{2}\sigma^2\right)t\right] \Big/ \sigma\sqrt{t} \tag{2}$$

In Part I we defined the variable p to be the cumulative probability of default at time t from the perspective of time zero. We defined the function $CND(Z)$ to be the cumulative normal distribution function for a normally-distributed random variable Z with mean zero and variance one. Using Equation (2) above we can make the following statement...

$$p=\int_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\delta Z=CND(a) \tag{3}$$

We will define the function $CND^{-1}(Z)$ to be the inverse of the cumulative normal distribution function for a normally-distributed random variable Z with mean zero and variance one. Using Equation (3) above we can define the default point by the following equation...

$$a=CN^{-1}(p) \tag{4}$$

Our first task will be to equate Equations (2) and (4) above as follows...

$$CND^{-1}(p)=\left[\ln\left(\frac{D_t}{A_0}\right)-\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right]\Big/\sigma\sqrt{t} \tag{5}$$

Using Equations (4) and (5) above we will define the following function of the actual, post calibration volatility parameter σ...

$$f(\sigma)=CND^{-1}(p) \tag{6}$$

Using Equations (4) and (5) above we will define the following function of the guess value of the volatility parameter $\hat{\sigma}$...

$$f(\hat{\sigma})=\left[\ln\left(\frac{D_t}{A_0}\right)-\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right]\Big/\sigma\sqrt{t} \tag{7}$$

The equation for the derivative of Equation (7) above with respect to $\hat{\sigma}$ is...

$$f'(\hat{\sigma})=\left\{(\hat{\sigma}t)(\hat{\sigma}\sqrt{t})-\sqrt{t}\left[\ln\left(\frac{D_T}{A_0}\right)-\left(\kappa-\phi-\frac{1}{2}\,\hat{\sigma}^2\right)t\right]\right\}\times\frac{1}{\hat{\sigma}^2 t} \tag{8}$$

Using Equations (6), (7) and (8) above we calculate the actual, post calibration value of σ by iterating the following equation (i.e. the Newton-Raphson method of solving nonlinear equations) until the error term ϵ goes to zero and $\sigma=\hat{\sigma}$... [3]

$$\hat{\sigma}+\frac{f(\sigma)-f(\hat{\sigma})}{f'(\hat{\sigma})}=\sigma+\epsilon \tag{9}$$

Model Calibration - Expected Recovery

In Part I we defined the variable π to be the recovery rate and the variable Γ to be the ratio of liquidation value to going-concern value. In Equation (2) above we defined the variable a to be the default point. Using Equation (1) above the equation for expected recovery is...

$$\begin{aligned}p\,\pi\,D_t&=\int_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\Gamma A_0+\sigma\sqrt{t}\,Z\Big\}\,\delta Z\\&=\Gamma A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right\}\int_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\left(Z^2-2\,\sigma\sqrt{t}\,Z\right)\right\}\delta Z\end{aligned} \tag{10}$$

We will make the following definitions...

$$\theta=Z-\sigma\sqrt{t}\ \text{ ...where... }\ \theta^2=Z^2-2\,\sigma\sqrt{t}\,Z+\sigma^2 t \tag{11}$$

Using the definitions in Equation (11) above we can rewrite Equation (10) above as...

$$\begin{aligned}p\,\pi\,D_t&=\Gamma A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right\}\int_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\mathrm{Exp}\left\{\frac{1}{2}\,\sigma^2 t\right\}\delta Z\\&=\Gamma A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi-\frac{1}{2}\,\sigma^2\right)t\right\}\mathrm{Exp}\left\{\frac{1}{2}\,\sigma^2 t\right\}\int_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta Z\\&=\Gamma A_0\,\mathrm{Exp}\left\{\left(\kappa-\phi\right)t\right\}\int_{-\infty}^{a}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta Z\end{aligned} \tag{12}$$

The equation for the derivative of Equation (11) above is...

$$\frac{\delta\theta}{\delta Z} = 1 \text{ ...such that... } \delta Z = \delta\theta \tag{13}$$

Using Equation (13) above we can rewrite Equation (12) above as...

$$\begin{aligned} p\,\pi\,D_t &= \Gamma A_0 \,\text{Exp}\left\{\left(\kappa - \phi\right)t\right\} \int_{-\infty-\sigma\sqrt{t}}^{a-\sigma\sqrt{t}} \sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta \\ &= \Gamma A_0 \,\text{Exp}\left\{\left(\kappa - \phi\right)t\right\} \int_{-\infty}^{a-\sigma\sqrt{t}} \sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,\theta^2\right\}\delta\theta \\ &= \Gamma A_0 \,\text{Exp}\left\{\left(\kappa - \phi\right)t\right\} CND\left[a - \sigma\sqrt{t}\right] \end{aligned} \tag{14}$$

Using Equation (14) above and solving for Γ we get...

$$\Gamma = p\,\pi\,D_t \Bigg/ \left(A_0 \,\text{Exp}\left\{\left(\kappa - \phi\right)t\right\} CND\left[a - \sigma\sqrt{t}\right]\right) \tag{15}$$

The Solution To Our Hypothetical Problem

Question 1: What is the value of the volatility parameter σ?

Using Equation (6) above and the parameters from Table 1 above the value of the normally-distributed random variable Z where asset value equals debt value (i.e. the default point) is...

$$f(\sigma) = CND^{-1}(0.1000) = -1.2816 \text{ Note: use Excel NORMSINV function} \tag{16}$$

Using Equation (7) above and the parameters from Table 1 above the equation for the iterated function $f(\hat{\sigma})$ given the value $\hat{\sigma}$ is...

$$f(\hat{\sigma}) = \left[\ln\left(\frac{500,000}{1,366,700}\right) - \left(0.0979 - 0.0732 - \frac{1}{2}\,\hat{\sigma}^2\right) \times 3.00\right] \Big/ \hat{\sigma}\,\sqrt{3.00} \tag{17}$$

Using Equation (8) above and the parameters from Table 1 above the equation for the iterated derivative of the function $f(\hat{\sigma})$ given the value $\hat{\sigma}$ is...

$$f'(\hat{\sigma}) = \left\{(\hat{\sigma} \times 3.00)(\hat{\sigma} \times \sqrt{3.00}) - \sqrt{3.00}\left[\ln\left(\frac{500,000}{1,366,700}\right) - \left(0.0979 - 0.0732 - \frac{1}{2}\,\hat{\sigma}^2\right) \times 3.00\right]\right\} \times \frac{1}{\hat{\sigma}^2 \times 3.00} \tag{18}$$

Using Equations (9), (16), (17) and (18) above and our guess value for $\hat{\sigma} = 0.10$, we can create the following table...

Iteration	Guess	$f(\sigma)$	$f(\hat{\sigma})$	$f'(\hat{\sigma})$
1	0.1000	-1.2816	-6.1465	63.1967
2	0.1770	-1.2816	-3.3686	20.7659
3	0.2775	-1.2816	-2.0060	8.9611
4	0.3583	-1.2816	-1.4292	5.7206
5	0.3841	-1.2816	-1.2900	5.0902
6	0.3858	-1.2816	-1.2816	5.0541
7	0.3858	-1.2816	-1.2816	5.0539
8	0.3858	-1.2816	-1.2816	5.0539

Answer To Question 1:

$$\text{Volatility parameter } \sigma = 0.3858 \tag{19}$$

Question 2: What is the value of the recovery given default parameter Γ?

The default point is the value of the random variable Z where enterprise value equals debt value. The default point (a) using the parameters from Table 1 above is...

$$
\begin{aligned}
a &= \left[\ln\left(\frac{D_t}{A_0}\right) - \left(\kappa - \phi - \frac{1}{2}\sigma^2\right)t\right] \Big/ \sigma\sqrt{t} \\
&= \left[\ln\left(\frac{500,000}{1,366,700}\right) - \left(0.0979 - 0.0732 - \frac{1}{2}0.3858^2\right) \times 3.00\right] \Big/ \left(0.3858 \times \sqrt{3.00}\right) \\
&= -1.2816
\end{aligned} \tag{20}
$$

Using Equation (20) above and the parameters from Table 1 above note the following equation...

$$
CND\left[a - \sigma\sqrt{t}\right] = CND\left[-1.2816 - 0.3858 \times \sqrt{3}\right] = NORMSDIST(-1.9498) = 0.0256 \tag{21}
$$

Using Equation (21) above and the parameters from Table 1 above the value of parameter Γ is...

$$
\begin{aligned}
\Gamma &= p\,\pi\,D_t \Big/ \left(A_0\,\mathrm{Exp}\left\{\left(\kappa - \phi\right)t\right\} CND\left[a - \sigma\sqrt{t}\right]\right) \\
&= 0.10 \times 0.40 \times 500,000 \Big/ \left(1,366,700 \times \mathrm{Exp}\left\{\left(0.0979 - 0.0732\right) \times 3\right\} \times 0.0256\right. \\
&= 0.5308
\end{aligned} \tag{22}
$$

References

[1] Gary Schurman, *Loan Guarantees: The Two State Default Model,* April, 2017.

[2] Gary Schurman, *Loan Guarantees: The Revised BSOPM - Model Basics,* May, 2017.

[3] Gary Schurman, *Loan Guarantees: The Newton-Raphson Method,* Oct, 2009.